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Symmetric function products and plethysms and the boson–fermion correspondence

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Abstract. We use the boson–fermion correspondence for S and Q functions to establish some interesting properties concerning outer products and plethysms of S-functions (or Q-functions) by power sum symmetric functions. The techniques which are developed are also applied to computing the inverse Kostka–Foulkes matrix (which is the transition matrix between Hall–Littlewood symmetric functions and S-functions) in some simple cases.

1. Introduction

The long-established boson–fermion correspondence [1] has been of great importance in many areas of mathematical physics. It has proved indispensable in vertex operator realizations of affine Lie algebras [2–5], which in turn can be used in the study of hierarchies of nonlinear partial differential equations [6, 7]. It has also proved useful in the theory of symmetric functions as it relates to these hierarchies. It has been used, for example, in the theory of the (B)KP hierarchy of nonlinear partial differential equations [8, 9] to prove that Schur (Q-)*polynomials* solve the hierarchy [10–13], as well as investigating the Hirota form of the (B)KP hierarchy [14]and its connection to the tensor product of two Fock representations of the Virasoro algebra. [15]. In addition it has proved helpful in proving S and Q-function identities [16–18], new determinant formulae for composite S and Q-functions [19], as well as investigating various q-deformations of symmetric functions [20].

It is in the spirit of the latter works that we approach the present work. There, the boson–fermion correspondence (along with Wicks theorem) was used as an algebraic tool to translate calculations involving free fermions (whose algebra has a very simple structure) into results concerning S and Q-functions. Indeed, one can generalize the correspondence to Hall–Littlewood symmetric functions [21] although, in this case, the structure of the corresponding fermionic algebra is not nearly so nice as that of free fermions. Nevertheless, as was shown in Jing's article, it is still possible to derive some powerful results concerning symmetric functions. By using the different realizations of the Heisenberg algebra in terms of power sum symmetric functions and in terms of a bilinear expression involving free fermions (neutral free fermions) we are able to find a method for decomposing the power sums $p_\lambda(x)$ in terms of S-functions (Q-functions). This enables one to calculate (albeit by brute force!) the characters χ^λ_μ of the symmetric group S_n which provide the transition between power sums and symmetric function

$$p_\lambda(x) = \sum_\mu \chi^\mu_\lambda \, s_\mu(x).$$

Using the same method, we also develop a very mechanical formula for the well known decomposition of the product of a power sum and an S-function, in terms of other S-functions. This implicitly carries the same information as the Murnaghan–Nakayama recurrence relation for the characters of the symmetric group [22].

We then turn our attention to computing plethysms [23, 24] of the form $s_\lambda(x) \otimes p_r(x)$. That is, we find a proceedure for decomposing $s_\lambda(x^r)$ in terms of S-functions $s_\mu(x)$, by relating the vertex operator realization of S-functions with argument x^r to those with argument x. This provides us with some exact results (particularly in the case $r = 2$) as well as providing an algorithm in the general case. We then turn our attention to Hall–Littlewood functions and in particular the inverse Kostka–Foulkes polynomials, which are the matrix elements of the transition matrix between Hall–Littlewood functions and S-functions. These polynomials have been calculated in terms of matrix elements of certain vertex operators [21], and there is indeed a direct combinatorial description of them [25, 26]. We use the techniques developed for the plethysm calculations to work out a method for calculating *inverse* Kostka–Foulkes polynomials, and derive a couple of explicit results.

In section 2 we briefly review the classical boson–fermion correspondence, before utilizing it to reproduce some well known identities involving power sums and S-functions. *Using the boson–fermion correspondence for Q-functions [10], these identities are extended* to that case in section 3. In section 4 we investigate the plethysms $s_\lambda(x) \otimes p_r(x) = s_\lambda(x^r)$. By manipulating the vertex operators representing the free fermionic currents, we are able to give a procedure for calculating this plethysm and write down some simple cases explicitly. Lastly, in section 5, we carry out similar manipulations to Hall–Littlewood vertex operators [21] to find an algorithm for calculating the decomposition of the Hall–Littlewood function $P_\lambda(x; t)$ in terms of S-functions. We conclude with some remarks about extending the concept of plethysm to Hall–Littlewood functions, and look at a couple of examples. In the appendix we display some interesting relationships between elementary Q-functions, the functions $h_n(x^2)$, and the compound functions $h_n(x, x)$.

2. S-functions and the boson–fermion correspondence

Let us summarize the boson–fermion correspondence for free fermions [8] which we shall be using extensively. The algebra $\mathcal{A}$ of free fermions is generated by ψ_i, ψ_i^*, $i \in \mathbb{Z}$ satisfying the anti-commutation relations

$$\{\psi_i, \psi_j\} = 0 = \{\psi_i^*, \psi_j^*\} \qquad \{\psi_i, \psi_j^*\} = \delta_{ij}. \tag{2.1}$$

There is a Fock representation of this algebra with a vacuum $|0\rangle$ which satisfies

$$\begin{aligned} &\psi_i|0\rangle = 0 \quad (i < 0) \qquad && \psi_i^*|0\rangle = 0 \quad (i \geqslant 0) \\ &\langle 0|\psi_i = 0 \quad (i \geqslant 0) \qquad && \langle 0|\psi_i^* = 0 \quad (i < 0). \end{aligned} \tag{2.2}$$

Using this definition of the vacuum, we can compute the vacuum expectation value $\langle a\rangle \equiv \langle 0|a|0\rangle$ for any product of free fermions. In particular we have

$$\langle \psi_i\psi_j\rangle = 0 \qquad\qquad \langle \psi_i^*\psi_j^*\rangle = 0$$

$$\langle \psi_i\psi_j^*\rangle = \begin{cases} \delta_{ij} & i = j < 0 \\ 0 & \text{otherwise} \end{cases} \qquad \langle \psi_i^*\psi_j\rangle = \begin{cases} \delta_{ij} & i = j \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

If we define normal ordering by $:\psi_i\psi_j^*: := \psi_i\psi_j^* - \langle\psi_i\psi_j^*\rangle$ then we have

$$:\psi_i\psi_j^*: := \begin{cases} \psi_i\psi_j^* & j \geqslant 0 \\ -\psi_j^*\psi_i & j < 0 \end{cases} \qquad :\psi_i^*\psi_j: := \begin{cases} \psi_i^*\psi_j & j < 0 \\ -\psi_j\psi_i^* & j \geqslant 0. \end{cases}$$

Let

$$H_n = \sum_{i\in\mathbb{Z}} :\psi_i\psi_{i+n}^*: \qquad n \in \mathbb{Z}. \tag{2.3}$$

Then the operators H_n generate the Heisenberg algebra

$$[H_n, H_m] = n\,\delta_{n+m,0}. \tag{2.4}$$

Suppose we have a set of Heisenberg generators $\{\alpha_n : n \in \mathbb{Z}\}$ satisfying (2.4). These have a realization on the space $\Lambda(x)$ of symmetric polynomials in the indeterminates $(x_1, x_2, \ldots)$ in terms of power sum symmetric functions $p_k(x) = \sum_i x_i^k$ as

$$\alpha_{-k} \leftrightarrow p_k \qquad \alpha_k \leftrightarrow k\frac{\partial}{\partial p_k} \quad \text{for } k > 0 \tag{2.5}$$

and α_0 acts as a constant on Λ. Let us adjoin to the Heisenberg algebra an operator q satisfying

$$[q, \alpha_n] = 0 \quad \text{for } n \neq 0 \qquad [q, \alpha_0] = i.$$

Define vertex operators

$$\begin{aligned} \psi(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{\alpha_n}{n}z^{-n}\right)\,\mathrm{e}^{iq}z^{\alpha_0}, \\ \psi^*(z) &= \exp\left(-\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}z^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{\alpha_n}{n}z^{-n}\right)\,z^{-\alpha_0}\mathrm{e}^{-iq}. \end{aligned} \tag{2.6}$$

If the modes of these vertex operators are given by the expansion

$$\psi(z) = \sum_{n\in\mathbb{Z}}\psi_n z^n \qquad \psi^*(z) = \sum_{n\in\mathbb{Z}}\psi_n^* z^{-n}$$

then it is well known that the modes ψ_n, ψ_n^* satisfy the anti-commutation relations of the free fermion algebra (2.1). Moreover, under the association (2.5) every state $a|0\rangle$, $a \in \mathcal{A}$ in the fermionic Fock space can be identified with a symmetric function as follows: Define a grading (charge) on the elements of $\mathcal{A}$ by setting $\deg(\psi_i) = 1$, $\deg(\psi_i^*) = -1$, for all $i \in \mathbb{Z}$ (this can be achieved by the grading operator $\mathrm{ad}(H_0)$ with H_0 defined in (2.3)). Thus an element $\psi_{j_1}^*\cdots\psi_{j_r}^*\psi_{i_1}\cdots\psi_{i_s}|0\rangle$ will have charge $l = s - r$. If we let

$$\langle l| = \begin{cases} \langle 0|\psi_{-1}\cdots\psi_l & \text{if } l < 0 \\ \langle 0| & \text{if } l = 0 \\ \langle 0|\psi_0^*\cdots\psi_{l-1}^* & \text{if } l > 0 \end{cases} \qquad H(x) = \sum_{n=1}^{\infty}\frac{1}{n}p_n(x)\,H_n$$

then we have the isomorphism of states

$$a|0\rangle \leftrightarrow \langle l|\mathrm{e}^{H(x)}a|0\rangle. \tag{2.7}$$

In fact each of the charge subspaces $\mathcal{A} = \oplus_l\mathcal{A}_l$ will be isomorphic to the space Λ of symmetric functions in the variables x. In particular there is the identification [8] for $0 \leqslant i_s < \cdots < i_1$, $0 < j_r < \cdots < j_1$

$$\psi_{-j_1}^*\cdots\psi_{-j_r}^*\psi_{i_s}\cdots\psi_{i_1}|0\rangle \;=\; (-1)^{j_1+\cdots j_r+l(l-1)/2}\,s_\lambda(x) \tag{2.8}$$

where

$$\lambda = (i_1 + 1 - l, i_2 + 2 - l, \ldots, i_s + s - l, r^{j_r - 1}, (r-1)^{j_{r-1} - j_r - 1}, \ldots, 2^{j_2 - j_3 - 1}, 1^{j_1 - j_2 - 1}). \tag{2.9}$$

Note that for $l = 0$, we can write this in Frobenius notation as

$$\lambda = \begin{pmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 - 1 & j_2 - 1 & \cdots & j_r - 1 \end{pmatrix} \equiv (i_1, i_2, \ldots, i_r | j_1 - 1, j_2 - 1, \ldots, j_r - 1).$$

The boson–fermion correspondence can be used to prove useful identities involving S-functions. For instance, we know that under the above isomorphism, the Heisenberg generators (2.3) are mapped onto the following operators on Λ.

$$H_{-n} \longleftrightarrow p_n(x) \qquad H_n \longleftrightarrow n \frac{\partial}{\partial p_n(x)} \quad n > 0 \tag{2.10}$$

while $H_0 \leftrightarrow 0$. Thus, acting on the vacuum, we know that $H_{-n}|0\rangle = p_n(x).1 = p_n(x)$, so that

$$\begin{aligned} p_n(x) = H_{-n}|0\rangle &= \sum_{i\in\mathbb{Z}} :\psi_i \psi^*_{i-n}: \ |0\rangle \\ &= \left(\sum_{i\geqslant n} \psi_i \psi^*_{i-n} - \left(\sum_{i<0} + \sum_{i=0}^{n-1} \right) \psi^*_{i-n} \psi_i \right) |0\rangle = \sum_{i=0}^{n-1} (-1)^j \, s_{(n-1-j|j)}(x) \end{aligned}$$

expressing the well known identity between power sums and one-hook S-functions. In a similar manner we can calculate $p_{(n,m)}(x) = p_n(x) p_m(x)$, by working out $H_{-n} H_{-m}|0\rangle$, and obtain

$$\begin{aligned} p_{(n,m)}(x) &= \left(\sum_{k=n+1}^{n+m} - \sum_{k=1}^{m} \right) (-1)^{n+m-k} \, s_{(k-1|n+m-k)}(x) \\ &\quad + \sum_{k=1}^{n} \sum_{l=1}^{m} (-1)^{n+m-k-l} \, s_{(k-1,l-1|n-k,m-l)}(x). \end{aligned}$$

In principle, one could expand the power sum $p_\lambda(x)$ in terms of S-functions by applying the operator $H_{-\lambda_1} H_{-\lambda_2} \cdots$ to the Fock space vacuum. However, it would be easier to use the following result recursively.

Lemma 1. If

$$\lambda = \begin{pmatrix} i_1 & \cdots & i_r \\ j_1 - 1 & \cdots & j_r - 1 \end{pmatrix}$$

where $j_k \geqslant 1$, $i_k \geqslant 0$, then

$$p_n \, s_\lambda = \sum_{q=1}^{r} \left(s_{\mu_q} - (-1)^n \, s_{\nu_q} \right) - \sum_{k=0}^{n-1} (-1)^{k-n} \, s_{\sigma_k}$$

where

$$\mu_q = \begin{pmatrix} i_1 & \cdots & i_q + n & \cdots & i_r \\ j_1 - 1 & \cdots & j_q - 1 & \cdots & j_r - 1 \end{pmatrix} \qquad \sigma_k = \left(\lambda \,\middle|\, \begin{matrix} k \\ n-k-1 \end{matrix} \right)$$

and

$$\nu_q = \begin{pmatrix} i_1 & \cdots & i_q & \cdots & i_r \\ j_1 - 1 & \cdots & j_q + n - 1 & \cdots & j_r - 1 \end{pmatrix}.$$

The proof uses the isomorphism (2.8) and formulae such as

$$\psi_j^* \psi_{i_1} \cdots \psi_{i_m} |0\rangle = \sum_{p=1}^{m} (-1)^{p-1} \delta_{j,i_p} \psi_{i_1} \cdots \widehat{\psi_{i_p}} \cdots \psi_{i_m} |0\rangle \qquad j \geqslant 0$$

where $\widehat{}$ denotes omission of the relevant object. Note that non-standard partitions μ_q, ν_q and σ_k will arise in the above expansion. But these are easily modified to standard partitions by noting that interchanging consecutive Frobenius labels introduces a minus sign in the S-function (so that, in particular if a partition has any two Frobenius labels equal, the corresponding S-function is zero). The above lemma was proved in [27] in the form

$$p_n\, s_\lambda = \sum_{\mu} (-1)^{\mathrm{ht}(\mu-\lambda)} s_\mu \tag{2.11}$$

where the sum is over all partitions μ such that the skew diagram $\theta = \mu - \lambda$ is a border strip of length n. By this we mean that θ is a connected skew diagram which contains no 2×2 blocks, the length of θ is $\sum_i \theta_i$, and $\mathrm{ht}(\theta)$ is the one less than the number of rows θ occupies. Hence to calculate $p_n\, s_\lambda$ one can use the algebraic result lemma 1, or the combinatorial result (2.11).

We can slightly generalize the Heisenberg generators (2.3) and in the process, obtain some interesting identities involving elementary Hall–Littlewood functions and S-functions. Let

$$H_n(t) = \sum_{i \in \mathbb{Z}} t^{-n-i} : \psi_i \psi_{i+n}^* : . \tag{2.12}$$

Then these generators fulfill the commutation relations

$$[H_n(t), H_m(t)] = (t^m - t^n) H_{n+m}(t^2) + t\, \frac{t^n - t^{-n}}{t - t^{-1}}\, \delta_{n+m,0}$$

so that the usual relations are obtained when $t = 1$. For generic t these generators do not form a closed algebra, although the generators $\{H_n(1), H_n(-1) : n \in \mathbb{Z}\}$ form an interesting $\mathbb{Z}_2$ graded algebra (although not a superalgebra). Also the set $\{H_n(-1) : n \text{ even}\}$ form an ordinary Heisenberg algebra and the set $\{H_n(-1) : n \text{ odd}\}$ an algebra of symplectic bosons [28]. The elegant representation of $H_n(1) \equiv H_n$ in terms of power sums and their adjoints unfortunately does not generalize to the $t \neq 1$ case. In fact, if we have the generating function $H(w) = \sum_n H_{-n} w^n$ then it follows that

$$H(w) =: \psi(w) \psi^*(tw) := Z(w, wt)$$

where [8]

$$Z(u, v) = \frac{v}{u - v} \left(\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n (u^n - v^n) \right) \exp\left(\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n} (u^{-n} - v^{-n}) \right) - 1 \right).$$

That is

$$\begin{aligned} H_0(t) &= \frac{t^{1-\alpha_0}}{1-t} \sum_{n=1}^{\infty} q_n(x;t) \bar{q}_n(x;t) \\ H_p(t) &= \frac{t^{1-\alpha_0}}{1-t} \sum_{n=0}^{\infty} q_n(x;t) \bar{q}_{n+p}(x;t) \end{aligned} \tag{2.13}$$

where $q_n(x;t)$ is an elementary Hall–Littlewood function (see section 5) and the differential operator $\bar{q}_n(x;t)$ has the generating function

$$\sum_{n=1}^{\infty} \bar{q}_n(x;t)\, z^n = \exp\left(\sum_{n=1}^{\infty}(t^{-n}-1)\frac{\partial}{\partial p_n(x)}\, z^n\right).$$

These have the usual limits (2.10) as $t \to 1$. Using (2.13) we see that

$$H_{-n}(t)|0\rangle = \frac{t}{1-t}\, q_n(x;t) = t\, P_{(n)}(x;t) \qquad n > 0$$

so that if we now use the definition (2.12), we have the well known result [27]

$$P_{(n)}(x;t) = \sum_{k=0}^{n-1}(-t)^k\, s_{(n-1-k|k)}(x) \tag{2.14}$$

where $P_\lambda(x;t)$ denotes the Hall–Littlewood function (see section 5 for the definition).

To see the action of $H_{-n}(t)H_{-m}(r)$ on the vacuum, we first need the result

$$\bar{q}_i(x,t)\, q_j(x;r) = \sum_{n=0}^{i}[n]_{r,t}\left\{q_{j-n}(x;r)\bar{q}_{i-n}(x;t) - \left(1+\frac{r}{t}\right) q_{j-n-1}(x;r)\bar{q}_{i-n-1}(x;t)\right.$$
$$\left. + \frac{r}{t}\, q_{j-n-2}(x;r)\bar{q}_{i-n-2}(x;t)\right\}$$

where $[n]_{r,t} = \frac{t^{-n}-tr^{n+1}}{1-tr}$. This is proved using the generating functions for q_n and $\bar{q}_n$. From this, one obtains the following two equivalent expressions for $H_{-n}(t)H_{-m}(r)|0\rangle$:

$$\left(tr\sum_{i=0}^{m}[i]_{r,t} - r(t+r)\sum_{i=1}^{m}[i-1]_{r,t} + r^2\sum_{i=2}^{m}[i-2]_{r,t}\right) P_{(i+n)}(x;t)\, P_{(m-i)}(x;r)$$
$$= \sum_{k=0}^{m-1} t^{n+m-k}(-r)^{m-k} s_{(k|n+m-k-1)}(x) - \sum_{k=0}^{m-1} t^k(-r)^{m-k} s_{(n+k|m-k-1)}(x)$$
$$+ \sum_{k=0}^{n-1}\sum_{l=0}^{m-1}(-t)^{n-k}(-r)^{m-l}\, s_{(k,l|n-k-1,m-l-1)}(x)$$

which is an unusual identity involving elementary Hall–Littlewood and S-functions. When $t = 1$ and $r = -1$ (or vice-versa), the left-hand side reduces to $-p_n(x)q_m(x)/2$ where $q_m(x) = \frac{1}{2}P_{(m)}(x;-1)$ is an elementary Q-function (see section 3), so we obtain the expansion of the product of a power sum and an elementary Q-function in terms of one and two-hook S-functions. This result could also be deduced however, by expanding $q_m(x)$ in terms of S-functions by (2.14) and using lemma 1.

3. Q-functions

Elementary Q-functions $q_n(x)$ are defined in terms of the generating function

$$\prod_i\left(\frac{1+x_i z}{1-x_i z}\right) = \sum_{n=1}^{\infty} q_n(x) z^n.$$

For a two part partition (m,n) (strict or non-strict), let

$$Q_{(m,n)}(x) = q_m(x)q_n(x) + 2\sum_{j=1}^{n}(-1)^j q_{m+j}(x)q_{n-j}(x).$$

Then $Q_{(m,n)}(x) = -Q_{(n,m)}(x)$ and so if $\lambda = (\lambda_1, \ldots, \lambda_p)$ is a partition with distinct parts, one can define

$$Q_\lambda(x) = \mathrm{Pf}\left(Q_{(\tilde{\lambda}_i, \tilde{\lambda}_j)}\right)$$

where $\tilde{\lambda} = \lambda$ if p is even, and $\tilde{\lambda} = (\lambda, 0)$ if p is odd. Here $\mathrm{Pf}(M)$ denotes the Pfaffian of the antisymmetric matrix M. There is likewise, an isomorphism between the Fock space generated by neutral free fermions, and Q-functions. The neutral free fermions ϕ_i, $i \in \mathbb{Z}$ are defined in terms of free fermions, by

$$\phi_j = \frac{\psi_j + (-1)^j \psi^*_{-j}}{\sqrt{2}}$$

which satisfy the anti-commutation relations

$$\{\phi_i, \phi_j\} = (-1)^i \delta_{i+j,0}. \tag{3.1}$$

There is a vacuum $|0\rangle$ defined by $\phi_i|0\rangle = 0$ for $i < 0$. Let

$$H_n = \tfrac{1}{2} \sum_{i \in \mathbb{Z}} (-1)^{i-1} : \phi_i \phi_{-i-n} : \tag{3.2}$$

with normal ordering defined as above. That is

$$: \phi_i \phi_j := \begin{cases} \phi_i \phi_j & \text{if } j < 0 \\ -\phi_j \phi_i & \text{if } j > 0 \\ (1 - \delta_{i,0}) \phi_i \phi_0 & \text{if } j = 0. \end{cases}$$

Then the generators $\{H_n : n \in 2\mathbb{Z} + 1\}$ generate the Heisenberg algebra

$$[H_n, H_m] = \frac{n}{2} \delta_{n+m,0}.$$

Again, if one lets

$$X(z) = \frac{1}{\sqrt{2}} \exp\left(2 \sum_{n \text{ odd}} \frac{p_n}{n} z^n\right) \exp\left(-\sum_{n \text{ odd}} \frac{\partial}{\partial p_n} z^{-n}\right) = \sum_{j \in \mathbb{Z}} X_j z^j$$

then there is an isomorphism $\phi_j \leftrightarrow X_j$. Moreover, there is an isomorphism of the states [11, 19]

$$\phi_{\lambda_1} \cdots \phi_{\lambda_p} |0\rangle \longleftrightarrow \begin{cases} \langle e^{H(x)} \phi_{\lambda_1} \phi_{\lambda_2} \cdots \phi_{\lambda_p} \rangle = 2^{-p/2} Q_\lambda(x) & \text{if } p \text{ is even} \\ \langle e^{H(x)} \phi_{\lambda_1} \phi_{\lambda_2} \cdots \phi_{\lambda_p} \phi_0 \rangle = 2^{-(p+1)/2} Q_\lambda(x) & \text{if } p \text{ is odd.} \end{cases}$$

In particular the state $\phi_m \phi_n |0\rangle \leftrightarrow \frac{1}{2} Q_{(m,n)}(x)$. Under this isomorphism, the Heisenberg generators have the realization

$$H_{-n} \longleftrightarrow p_n(x) \qquad H_n \longleftrightarrow \frac{n}{2} \frac{\partial}{\partial p_n(x)} \qquad n > 0, n \text{ odd}. \tag{3.3}$$

Like the S-function case, there is a relation between (odd) power sums and two-part Q-functions

$$p_{2k+1}(x) = \tfrac{1}{2} \sum_{j=0}^{k} (-1)^j Q_{(2k+1-j,j)}(x)$$

which is proved in a similar manner, using (3.2) and (3.3) Note that it is only possible to express the odd power sums in terms of Q-functions, because the space spanned by the Q-functions is isomorphic to $\mathbb{Q}[p_1, p_3, p_5, \ldots]$. Similarly we have

$$p_{(2m+1,2n+1)}(x) = \tfrac{1}{2}\sum_{j=0}^{n}(-1)^j Q_{(2m+2n+2-j,j)}(x) - \tfrac{1}{2}\sum_{j=1}^{n}(-1)^j Q_{(2m+1+j,2n+1-j)}(x)$$

$$+\tfrac{1}{4}\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j} Q_{(2m+1-i,i,2n+1-j,j)}(x).$$

Again, we can recursively obtain the decomposition of $p_\lambda(x)$, with each λ_i odd, by using

$$p_{2k+1}\, Q_{(\lambda_1,\ldots,\lambda_n)} = \sum_{j=1}^{n} Q_{(\lambda_1,\ldots,\lambda_j+2k+1,\ldots,\lambda_n)} + \frac{1}{2}\sum_{i=0}^{k}(-1)^i\, Q_{(\lambda_1,\ldots,\lambda_n,2k+1-i,i)} \tag{3.4}$$

the proof of which involves use of the anti-commutation relations of the neutral free fermions (3.1). Once again, the partitions in the above expression are non-standard and are changed to standard ones by noting that the interchange of any two consecutive partition labels introduces a minus sign in front of the Q-function. As in the S-function case, there is a combinatorial version of (3.4) the details of which we refer the reader to [29, 30].

Example. Using the elementary Q-functions

$$q_1 = 2p_1 \qquad q_2 = 2p_1^2 \qquad q_3 = \tfrac{4}{3}p_1^3 + \tfrac{2}{3}p_3 \qquad q_4 = \tfrac{2}{3}p_1^4 + \tfrac{4}{3}p_3p_1$$

$$q_5 = \tfrac{4}{15}p_1^5 + \tfrac{4}{3}p_3p_1^2 + \tfrac{2}{5}p_5 \qquad q_6 = \tfrac{4}{45}p_1^6 + \tfrac{8}{9}p_3p_1^3 + \tfrac{2}{9}p_3^2 + \tfrac{4}{5}p_5p_1$$

we have

$$p_3Q_{(2,1)} = \tfrac{4}{3}(p_3p_1^3 - p_3^2) = Q_{(5,1)} - Q_{(4,2)} + \tfrac{1}{2}Q_{(3,2,1)}.$$

4. S-function plethysms

In this section we will examine (outer) plethysms of the type $s_\lambda \otimes p_r = s_\lambda(x_1^r, x_2^r, \ldots)$, and use these results to show how one could calculate some more general types of plethysms. Many explicit results concerning plethysms are known [31, 32], along with lots of powerful theorems which aid in their evaluation [23, 24]. We shall, as a contrast, use the boson–fermion correspondence as a tool for the decomposition of $s_\lambda(x^r)$ in terms of S-functions with argument x, which will enable us to calculate other plethysms direct from the definition.

Let us examine the case $r = 2$ in detail. We know that

$$\sum_{n=0}^{\infty} h_n(x^2)z^{2n} = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}p_{2n}(x)z^{2n}\right) = \exp\left(\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n}p_n(x)z^n\right)$$

$$= \left(\sum_{k=0}^{\infty} h_k(x)z^k\right)\left(\sum_{l=0}^{\infty}(-1)^l h_l(x)z^l\right)$$

so that

$$h_n(x^2) = \sum_{k=0}^{2n}(-1)^k h_k(x)h_{2n-k}(x).$$

Using

$$h_k h_{2n-k} = \sum_{j=0}^{k} s_{(2n-j,j)} \tag{4.1}$$

and rearranging the sum, we obtain the result [33]

$$h_n(x^2) = \sum_{j=0}^{n} (-1)^j \, s_{(2n-j,j)}(x) \tag{4.2}$$

along with the conjugate relation

$$e_n(x^2) = \sum_{j=0}^{n} (-1)^j \, s_{(2j,1^{2n-2j})}(x).$$

To calculate $s_\lambda(x^2)$ for general λ, we proceed as follows. Let

$$\begin{aligned}\Psi(z) &= \exp\left(2{\sum_n}' \frac{p_n(x)}{n} z^n\right) \exp\left(-{\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right) e^{iq} z^{2\alpha_0} \\ \Psi^*(z) &= \exp\left(-2{\sum_n}' \frac{p_n(x)}{n} z^n\right) \exp\left({\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right) z^{-2\alpha_0} e^{-iq}\end{aligned} \tag{4.3}$$

where ${\sum_n}' \equiv \sum_{n \text{ even}}$. Note that the vertex operators (4.3) are just the operators (2.6) with $x \to x^2$ and $z \to z^2$. Thus if we write $\Psi(z) = \sum_{n\in\mathbb{Z}} \Psi_n z^{2n}$, $\Psi^*(z) = \sum_{n\in\mathbb{Z}} \Psi_n^* z^{-2n}$, then

$$\Psi^*_{-j_1} \cdots \Psi^*_{-j_r} \Psi_{i_r} \cdots \Psi_{i_1} |0\rangle \;\longleftrightarrow\; (-1)^{j_1+\cdots j_r + l(l-1)/2} \, s_\lambda(x^2)$$

where λ is given by (2.9). Now, let $\widehat{\psi}(z) = \psi(z)\psi(-z)$, $\widehat{\psi^*}(z) = \psi^*(z)\psi^*(-z)$. That is

$$\widehat{\psi}(z) = 2 \; \exp\left(2{\sum_n}' \frac{p_n(x)}{n} z^n\right) \exp\left(-2{\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right) e^{2iq} (-1)^{\alpha_0} z^{2\alpha_0+1}$$

$$\widehat{\psi^*}(z) = 2 \; \exp\left(-2{\sum_n}' \frac{p_n(x)}{n} z^n\right) \exp\left(2{\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right) (-1)^{\alpha_0-1} z^{-2\alpha_0-1} e^{-2iq}.$$

Then $\Psi(z) = \widehat{\psi}(z)\xi(z)$ where

$$\xi(z) = \tfrac{1}{2} z \, e^{-iq} (-1)^{\alpha_0-1} \exp\left({\sum_n}' \frac{\partial}{\partial p_n(x)} z^{-n}\right).$$

We can expand $\widehat{\psi}(z)$ as a power series $\widehat{\psi}(z) = \sum_{n\in\mathbb{Z}} \widehat{\psi}_n z^{2n-1}$, where

$$\widehat{\psi}_n = \sum_{j\in\mathbb{Z}} (-1)^j \psi_{2n-1}\psi_j = 2 \sum_{j\leqslant n-1} (-1)^j \psi_{2n-1}\psi_j. \tag{4.4}$$

If we now use the fact that $h_n(x^2) = \Psi_n|0\rangle$, we have (ignoring the factors of $2\pi \mathrm{i}$ in this and subsequent integrals)

$$\begin{aligned} h_n(x^2) &= \int \frac{dz}{z} z^{-2n} \Psi(z)|0\rangle = -\tfrac{1}{2} \int \frac{dz}{z} z^{-2n+1} \widehat{\psi}(z) \, e^{-iq} |0\rangle \\ &= -\sum_{j=-1}^{n-1} (-1)^j \psi_{2n-1-j}\psi_j e^{-iq} |0\rangle . \end{aligned}$$

Upon using the fact that $\psi_i\psi_j e^{kiq}|0\rangle = s_{(i-k-1,j-k)}(x)$, we recover (4.2). In a similar manner we can consider $s_{(n-1,m)}(x^2) = \Psi_n\Psi_m|0\rangle$, and get

$$\Psi_n\Psi_m|0\rangle = \int \frac{\mathrm{d}z}{z} z_1^{-2n} z_2^{-2m} \widehat{\psi}(z_1)\xi(z_1)\widehat{\psi}(z_2)\xi(z_2)|0\rangle$$

$$= \sum_{p=-1}^{m-1} \widehat{\psi}_{n+m-1-p}\widehat{\psi}_p\, \mathrm{e}^{-2iq}|0\rangle$$

where we have used the property $\xi(z)\widehat{\psi}(w) = -w^2(1-w^2/z^2)^{-1}\widehat{\psi}(w)\xi(z)$. Hence

$$s_{(n-1,m)}(x^2) = \sum_{p=0}^{m}\sum_{j=-2}^{n+p-1}\sum_{k=-2}^{m-2-p} (-1)^{j+k} s_{(k-1,2m-2p-3-k,j+1,2n+2p+1-j)}(x). \tag{4.5}$$

As for the results of section 2, the partitions that occur on the right-hand side of (4.5) may be non-standard, and hence must be modified using the standard rules. For the general case we have

$$s_{(n_1-p+1,n_2-p+2,\ldots,n_{p-1}-1,n_p)}(x^2)$$

$$= \Psi_{n_1}\Psi_{n_2}\cdots\Psi_{n_p}|0\rangle$$

$$= (-1)^p \sum_{k_1,\ldots,k_p} \int \frac{\mathrm{d}z}{z} z_1^{2(k_1-n_1)} z_2^{2(k_2-n_2+1)} \cdots z_p^{2(k_p-n_p+p-1)}$$

$$\times \prod_{i<j}\left(1-\frac{z_j^2}{z_i^2}\right)\widehat{\psi}_{k_1}\cdots\widehat{\psi}_{k_p}\, \mathrm{e}^{-piq}|0\rangle$$

$$= (-1)^p \prod_{i<j}\left(1-R_{ij}\right)^{-1}\widehat{\psi}_{n_1}\widehat{\psi}_{n_2-1}\cdots\widehat{\psi}_{n_p-p+1}\, \mathrm{e}^{-piq}|0\rangle \tag{4.6}$$

where R_{ij} acts as a raising operator:

$$R_{ij}\widehat{\psi}_{\lambda_1}\cdots\widehat{\psi}_{\lambda_i}\cdots\widehat{\psi}_{\lambda_j}\cdots\widehat{\psi}_{\lambda_p} = \widehat{\psi}_{\lambda_1}\cdots\widehat{\psi}_{\lambda_i+1}\cdots\widehat{\psi}_{\lambda_j-1}\cdots\widehat{\psi}_{\lambda_p}.$$

One can now use (4.4) to rewrite (4.6) in terms of S-functions of the argument x. For partitions of length greater than two, the results are not aesthetically appealing so we do not bother to write them down explicitly.

There is, however, a neat formula for one-hook S-functions with argument x^2. Recall that $\Psi^*_{-j}\Psi_i|0\rangle = (-1)^j s_{(i|j-1)}(x^2)$. Again, we can write $\Psi^*(z) = \widehat{\psi^*}(z)\xi^*(z)$ where

$$\xi^*(z) = (-1)^{\alpha_0+1} z e^{iq} \exp\left(-\sum_n{}' \frac{\partial}{\partial p_n(x)} z^{-n}\right).$$

Now $\xi^*(z)\widehat{\psi}(w) = (z^{-2}-w^{-2})\widehat{\psi}(w)\xi^*(z)$, so we get

$$(-1)^j s_{(i|j-1)}(x^2) = \sum_{p=1}^{j}\sum_{q=0}^{i-1}(-1)^{p+q+1} s_{(q,2i-1-q|2j-p,p-1)}(x) + s_{(2i|2j-1)}(x)$$

$$+ \sum_{p=1}^{j-1}\sum_{q=0}^{i}(-1)^{p+q} s_{(q,2i+1-q|2j-2-p,p-1)}(x) - s_{(2i+1|2j-2)}(x).$$

4.1. The general case

The extension to calculating $s_\lambda(x^r)$ proceeds as follows: Let ω be a primitive rth root of unity. That is $\omega^r = 1$ and

$$1 + \omega^k + \omega^{2k} + \cdots + \omega^{(r-1)k} = \begin{cases} r & k \equiv 0, \quad (\text{mod } r) \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\sum_n{}'' \equiv \sum_{n\equiv 0\,(\text{mod } r)}$$

and

$$\Psi(z) = \exp\left(r\sum_n{}'' \frac{p_n(x)}{n} z^n\right) \exp\left(-\sum_n{}'' \frac{\partial}{\partial p_n(x)} z^{-n}\right) e^{iq} z^{r\alpha_0}.$$

Then if $\Psi(z) = \sum_{n\in\mathbb{Z}} \Psi_n z^{rn}$, we have

$$\Psi_{i_1} \cdots \Psi_{i_s}|0\rangle \longleftrightarrow s_\lambda(x^r)$$

where $\lambda = (i_1 - s + 1, i_2 - s + 2, \ldots, i_s)$. Again, let

$$\widehat{\psi}(z) = \psi(z)\psi(\omega z) \cdots \psi(\omega^{r-1} z)$$

$$= \prod_{j=1}^{r-1} \left((1-\omega^j)^{n-j} \exp\left(r\sum_n{}'' \frac{p_n(x)}{n} z^n\right) \exp\left(-r\sum_n{}'' \frac{\partial}{\partial p_n(x)} z^{-n}\right)\right.$$

$$\left. \times\, e^{riq}\, z^{r\alpha_0 + r(r-1)/2}\, \omega^{r(r-1)\alpha_0/2 + r(r-1)(r-2)/6}\right).$$

Then $\Psi(z) = \widehat{\psi}(z)\xi(z)$ where

$$\xi(z) = \prod_{j=1}^{r-1} \left((1-\omega^j)^{j-n}\, z^{r(r-1)/2}\, \omega^{r(r-1)(2r-1)/6}\, e^{-(r-1)iq}\, \omega^{-r(r-1)\alpha_0/2}\right.$$

$$\left. \times \exp\left((r-1)\sum_n{}'' \frac{\partial}{\partial p_n(x)} z^{-n}\right)\right).$$

If we expand

$$\widehat{\psi}(z) = \sum_{p\in\mathbb{Z}} \widehat{\psi}_p\, z^{rp - r(r-1)/2}$$

then

$$\widehat{\psi}_p = \sum_{\substack{i_1, i_2, \ldots, i_r \\ i_1 + \cdots + i_r = rp - r(r-1)/2}} \omega^{i_2 + 2i_3 + \cdots + (r-1)i_r} \psi_{i_1} \cdots \psi_{i_r}$$

$$= r \sum_{\substack{i_1 < i_2 < \ldots < i_r \\ i_1 + \cdots + i_r = rp - r(r-1)/2}} \left(\sum_{\sigma\in S_{r-1}} (\operatorname{sgn}\sigma)\, \omega^{i_{\sigma(2)} + 2i_{\sigma(3)} + \cdots + (r-1)i_{\sigma(r)}}\right) \psi_{i_1} \cdots \psi_{i_r} \tag{4.7}$$

where S_{r-1} denotes the symmetric group on the elements $\{2, 3, \ldots, r\}$. From here on, we can simply follow the $r = 2$ case. For example we have

$$\Psi_n|0\rangle = \int \frac{dz}{z} z^{-rn}\widehat{\psi}(z)\xi(z)|0\rangle = \frac{\omega^{r(r-1)(2r-1)/6}}{\prod_{j=1}^{r-1}(1-\omega^j)^{n-j}}\widehat{\psi}_n \, e^{-(r-1)iq}|0\rangle . \quad (4.8)$$

Using

$$\psi_{i_1}\cdots\psi_{i_r} e^{ikq}|0\rangle = s_{(i_1-(r-1)-k, i_2-(r-2)-k, \ldots, i_r-k)}(x)$$

we are able to express $h_n(x^r)$ in terms of S-functions with argument x. Because the above construction mirrors the $r = 2$ case, one can similarly write down an expression for $s_\lambda(x^r)$, for general λ.

Example. In the case $r = 3$, with $\omega^3 = 1$, using (4.7) and (4.8) we have

$$h_n(x^3) = \frac{1}{\omega^2-\omega} \sum_{\substack{-2\leqslant i_1<i_2<i_3 \\ i_1+i_2+i_3=3n-3}} \left(\omega^{i_2+2i_3} - \omega^{i_3+2i_2}\right) s_{(i_3, i_2+1, i_1+2)}(x) \quad (4.9)$$

so that, for example

$$h_3(x^3) = s_{(9)}(x) - s_{(81)}(x) + s_{(711)}(x) + s_{(63)}(x) - s_{(621)}(x) - s_{(54)}(x) + s_{(522)}(x)$$
$$+ s_{(441)}(x) - s_{(432)}(x) + s_{(333)}(x)$$

which may be checked explicitly by noting that, in terms of power sums, both sides are equal to $(p_3^3 + 3p_6p_3 + 2p_9)/6$. It is interesting to note that not all of the terms on the right-hand side of (4.9) are non-trivial (e.g. the ones corresponding to $(i_1, i_2, i_3) = (-2, 1, 7)$ or $(-1, 2, 5)$).

Using the above results we are in a position to calculate some plethysms by brute force (from the definition). Recall that, to calculate $s_\lambda \otimes s_\mu$, one expresses $s_\mu(x)$ as a multinomial in the power sums $p_1(x)$, $p_2(x)$, ... and then make the substitution $p_j(x) \to s_\lambda(x^j)$. Thus, for example, to calculate $h_n \otimes h_2$, write $h_2 = \frac{1}{2}(p_2 + p_1^2)$, so that

$$h_n \otimes h_2 = \tfrac{1}{2}\left(h_n(x^2) + h_n^2(x)\right).$$

Upon using (4.1) and (4.2), we recover the well known results

$$\begin{aligned} s_{(n)} \otimes s_{(2)} &= \sum_{j=0}^{[n/2]} s_{(2n-2j, 2j)} \\ s_{(n)} \otimes s_{(1^2)} &= \sum_{j=1}^{[(n+1)/2]} s_{(2n-2j+1, 2j-1)}. \end{aligned} \quad (4.10)$$

Conversely if one knew that (4.10) were true, one could use the right-distributive law for plethysm, to calculate $s_{(n)} \otimes p_2 = s_{(n)} \otimes (s_{(2)} - s_{(1^2)})$, which was the method used in [33] to prove (4.2).

One could go on to calculate $s_{(n)} \otimes s_{(3)}$ from the definition

$$s_{(n)} \otimes s_{(3)} = \tfrac{1}{3}h_n(x^3) + \tfrac{1}{2}h_n(x^2)h_n(x) + \tfrac{1}{6}(h_n(x))^3$$

by using (4.1), (4.2) and (4.9) along with

$$s_{(2n-j,j)}(x)\, h_n = \sum_{p=0}^{n} \sum_{q=\max(0,n-j-p)}^{\min(n-p,2n-2j)} s_{(2n-j+p, j+q, n-p-q)}(x).$$

In a similar manner, one can obtain explicit expressions for the plethysms $s_{(n)} \otimes s_{(21)}$ and $s_{(n)} \otimes s_{(1^3)}$ using $s_{(21)} = (p_1^3 - p_3)/3$ and $s_{(1^3)} = p_1^3/6 - p_2p_1/2 + p_3/3$.

5. Hall–Littlewood functions

Finally, there is the boson–fermion correspondence between Hall–Littlewood symmetric functions, and the generalized fermions of Jing [21]. Instead of the vertex operators used there, we use a slightly shifted version which reduce to the usual ones when $t = 0$,

$$\varphi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{n} p_n(x) z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)} z^{-n}\right) e^{iq} z^{\alpha_0}$$

$$\varphi^*(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^n}{n} p_n(x) z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{\partial}{\partial p_n(x)} z^{-n}\right) z^{-\alpha_0} e^{-iq}$$

the components of which, obey the anti-commutation relations

$$\{\varphi_n, \varphi_m\} = t\varphi_{n+1}\varphi_{m-1} + t\varphi_{m+1}\varphi_{n-1}$$

$$\{\varphi_n^*, \varphi_m^*\} = t\varphi_{n-1}^*\varphi_{m+1}^* + t\varphi_{m-1}^*\varphi_{n+1}^*$$

$$\{\varphi_n, \varphi_m^*\} = t\varphi_{n-1}\varphi_{m-1}^* + t\varphi_{m+1}^*\varphi_{n+1} + (1-t)^2\delta_{n,m}.$$

If the bra and ket vacua are defined as in (2.2), then we have the equivalence [21]

$$\varphi_{\lambda_1}\varphi_{\lambda_2}\cdots\varphi_{\lambda_{p-1}}\varphi_{\lambda_p}|0\rangle \longleftrightarrow Q_\lambda(x;t)$$

where $\lambda = (\lambda_1 - p + 1, \lambda_2 - p + 2, \ldots, \lambda_p)$. It is not altogether obvious to what function the state $\varphi^*_{-j_1}\cdots\varphi^*_{-j_r}\varphi_{i_s}\cdots\varphi_{i_1}|0\rangle$ is mapped. For the simplest state $\varphi^*_{-j}\varphi_i|0\rangle$, we have

$$\varphi^*_{-j}\varphi_i|0\rangle \longleftrightarrow f_{i,j}(x;t) = q_j(0/x;t)q_i(x;t) + (1-t)\sum_{k=1}^{i} q_{j+k}(0/x;t)q_{i-k}(x;t)$$

where the supersymmetric functions $q_n(x/y;t)$ have the generating function (see [34])

$$\sum_{n=1}^{\infty} q_n(x/y;t)z^n = \prod_i \left(\frac{1-tx_iz}{1-x_iz}\right)\left(\frac{1-y_iz}{1-ty_iz}\right).$$

In particular we have $q_n(0/x;t) = (-1)^n S_{(1^n)}(x;t)$, where $S_\lambda(x;t) = \det(q_{\lambda_i-i+j}(x;t))$. The function $f_{i,j}(x;t)$ reduces down to the one-hook S-function $s_{(i|j-1)}(x)$ in the limit $t \to 0$, but does *not* represent a one-hook Hall–Littlewood function, which can be expressed as [35]

$$Q_{(n-k,1^k)}(x;t) = (1-t)(1-t^2)\cdots(1-t^k)\sum_{j=0}^{k}(-1)^j q_{n+j}(x;t)\, e_{k-j}(x)$$

where $e_n(x)$ is the nth elementary symmetric function. In fact, since $q_n(0/x;-1) = (-1)^n q_n(x;-1)$, then $f_{i,j}(x;-1) = Q_{(i,j)}(x)$, a two-part Q-function. From the relation

$$\sum_{n=0}^{p} q_n(x;t)q_{p-n}(0/x;t) = 0$$

we see that

$$f_{i,j}(x;t) = (t+1)\, q_i(x;t)\, q_j(0/x;t) - f_{j,i}(0/x;t)$$

so that when $t = -1$, we get $Q_{(i,j)}(x) = -Q_{(j,i)}(x)$ as per usual. Because we do not know how to express $\varphi^*_{-j_1}\cdots\varphi^*_{-j_r}\varphi_{i_s}\cdots\varphi_{i_1}|0\rangle$ in terms of Hall–Littlewood functions, it appears that we are unable to calculate the product of a Hall–Littlewood function and a power sum. One way to get around this is to express the Hall–Littlewood function $P_\lambda(x;t) \equiv b_\lambda^{-1}(t) Q_\lambda(x;t)$, where

$$b_\lambda(t) = \prod_i \gamma_{m_i(\lambda)}(t) \qquad \gamma_j(t) = (1-t)(1-t^2)\cdots(1-t^j)$$

and $m_i(\lambda)$ is the number of times i occurs in the partition λ, in terms of S-functions

$$P_\lambda(x;t) = \sum_\mu K^{-1}_{\lambda\mu}(t)\, s_\mu(x) \tag{5.1}$$

using the inverse Kostka–Foulkes matrix $K^{-1}_{\lambda\mu}(t)$, use lemma 1, and then re-express the the result in terms of the $P_\lambda(x;t)$ using $K_{\lambda\mu}(t)$. As previously mentioned, there are quite a number of explicit results concerning Kostka–Foulkes matrices, and it is our intention to now describe a way of calculating their inverses via (5.1).

It turns out that we can use the techniques of section 4, where we decompose a vertex operator in terms of products of free fermionic currents and an annihilation part, to find a simple way of calculating the inverse Kostka–Foulkes matrix elements $K^{-1}_{\lambda\mu}(t)$. Write $\varphi(z) = \widehat{\varphi}(z)\,\xi(z)$ where $\widehat{\varphi}(z) = \psi(z)\psi^*(tz)$ and

$$\xi(z) = (1-t)t^{\alpha_0 - 1} e^{iq} z^{\alpha_0} \exp\left(-\sum_{n=1}^\infty t^{-n} \frac{\partial}{\partial p_n(x)} z^{-n}\right).$$

Note that we can write $\widehat{\varphi}(z) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}_n z^n$, where

$$\widehat{\varphi}_n = \sum_{j\in\mathbb{Z}} t^j\, \psi_{n-j}\psi^*_{-j}. \tag{5.2}$$

Thus

$$\begin{aligned} Q_{(n)}(x;t) &= \varphi_n|0\rangle = (1-t)\,\widehat{\varphi}_n\, e^{iq}|0\rangle \\ &= (1-t)\sum_{j=0}^{n-1} t^j\, \psi_{n-j}\psi^*_{-j}\, e^{iq}|0\rangle \end{aligned}$$

and we recover equation (2.14). Similarly, we can use the fact that

$$\xi(z)\widehat{\varphi}(w) = \frac{tz-w}{z-w}\,\widehat{\varphi}(w)\xi(z)$$

to obtain

$$\varphi_n\varphi_m|0\rangle = t^2(1-t)^2\left(\widehat{\varphi}_{n-1}\widehat{\varphi}_m + (1-t^{-1})\sum_{j=0}^{m-1}\widehat{\varphi}_{n+j}\widehat{\varphi}_{m-j-1}\right)e^{2iq}|0\rangle$$

so that upon using (5.2), as well as

$$\widehat{\varphi}_0\, e^{2iq}|0\rangle = \frac{t^{-1}}{1-t}\, e^{2iq}|0\rangle$$

we have

$$\begin{aligned} Q_{(n-1,m)}(x;t) = (1-t)^2 &\left\{ \sum_{q=-1}^{n+m-3} (-t)^q\, s_{(n+m-3-p|p+1)}(x) \right. \\ &+ \left(\sum_{j=-1}^{-1} + (1-t^{-1})\sum_{j=0}^{m-2}\right)\left(\sum_{q=-1}^{m-j-3}(-1)^{n+j+q} t^{2q+n+j+2}\, s_{(m-j-3-q|n+j+1+q)}(x)\right) \end{aligned}$$

$$- \sum_{q=-1}^{m-j-3} (-1)^q t^{2q+j+3-m} \, s_{(n+m-3-q|q+1)}(x)$$

$$\left. \left. - \sum_{p=-1}^{n+j-2} \sum_{q=-1}^{m-j-3} (-t)^{p+q+2} s_{(m-j-3-q,n+j-2-p|p+1,q+1)}(x) \right) \right\} . \tag{5.3}$$

Thus we are able to express two-part Hall–Littlewood functions in terms of one and two-hook S-functions. Again there is overcounting in the above equation, but this might be expected due to the complicated nature of inverse Kostka–Foulkes matrices.

Example. Using $n = 4$ and $m = 1$ in (5.3) we have

$$P_{(31)}(x;t) = \frac{1}{(1-t)^2} Q_{(31)}(x;t) = s_{(31)}(x) - t\, s_{(2^2)}(x) - t\, s_{(21^2)}(x) + (t^2 + t^3)\, s_{(1^4)}(x)$$

while for $n = 3$, $m = 2$, we get

$$P_{(22)}(x;t) = \frac{1}{(1-t)(1-t^2)} Q_{(22)}(x;t) = s_{(2^2)}(x) - t\, s_{(21^2)}(x) + t^3\, s_{(1^4)}(x).$$

For the general case we have

$$\varphi_{n_1} \cdots \varphi_{n_p} |0\rangle = (1-t)^p \, t^{p(p-1)} \prod_{i<j} \frac{1 - t^{-1} R_{ij}}{1 - R_{ij}} \, \widehat{\varphi}_{n_1 - p+1} \cdots \widehat{\varphi}_{n_p} \, e^{ipq} |0\rangle$$

and so application of (5.2) will allow one to express $P_\lambda(x;t)$ where λ is a p part partition in terms of one, two, up to p-hook S-functions.

6. Conclusions

We have used the (generalized) boson–fermion correspondence as a means of generating interesting identities amongst power sums, S-, Q- and Hall–Littlewood symmetric functions. We developed methods for decomposing $s_\lambda(x^r)$ which enabled us to examine some S-function plethysms directly from the definition. Finally we we able to apply similar methods for expanding Hall–Littlewood functions in terms of ordinary S-functions. It would seem that the methods used here could be applied to any symmetric function which is expressable in terms of the product of modes of a vertex operator, for example Milne's symmetric functions $H_\lambda(x;q)$ [36, 37]. Indeed, if a vertex operator realization of Macdonald's [38] generalized symmetric functions $P_\lambda(x;q,t)$ were available (of which the author remains unaware), these methods could be of help in studying the properties of these important functions.

One possible avenue for further investigation is to consider extending the definition of outer plethysm to Hall–Littlewood functions. For example, using

$$q_{n-k} q_k = Q_{(n-k,k)} + (1-t) \sum_{j=0}^{k-1} Q_{(n-j,j)}$$

we can mirror the derivation of $h_n(x^2)$ to show that

$$q_n(x^2;t^2) = (1+t) \sum_{j=0}^{n-1} (-1)^j Q_{(2n-j,j)}(x;t) + (-1)^n Q_{(n,n)}(x;t).$$

Thus if we define the plethysm $Q_\lambda \otimes Q_\mu$ to mean: express Q_μ as a multinomial in power sums and then make the substitution $p_j(x) \to Q_\lambda(x^j; t^j)$, then we have the results

$$q_n \otimes s_{(2)} = \begin{cases} Q_{(2n)} - tQ_{(2n-1,1)} + Q_{(2n-2,2)} - \cdots - tQ_{(n+1,n-1)} + Q_{(n,n)} & \text{if } n \text{ is even} \\ Q_{(2n)} - tQ_{(2n-1,1)} + Q_{(2n-2,2)} - \cdots + Q_{(n+1,n-1)} & \text{if } n \text{ is odd.} \end{cases}$$

and

$$q_n \otimes q_2 = \begin{cases} a(t)Q_{(2n)} + b(t)Q_{(2n-1,1)} + a(t)Q_{(2n-2,2)} & \\ \quad + \cdots + b(t)Q_{(n+1,n-1)} + (1-t)Q_{(n,n)} & \text{if } n \text{ is even} \\ Q_{(2n)} + b(t)Q_{(2n-1,1)} + a(t)Q_{(2n-2,2)} & \\ \quad + \cdots + b(t)Q_{(n+1,n-1)} + t(t-1)Q_{(n,n)} & \text{if } n \text{ is odd.} \end{cases}$$

where $a(t) = (1-t)(1+t^2)$ and $b(t) = 2t(t-1)$. There is no *a priori* reason why one should use t^j in the above definition of Hall–Littlewood plethysm, however it seems that the results are much nicer if this is the case. In particular, in the two examples above, the coefficients in the plethysm are elements of $\mathbb{Z}[t]$. It would be interesting to find out if this were true in general.

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Appendix. Determinant formulae for $h_n(x^2)$

There is an interesting relation between $h_n(x^2)$, the elementary Q-functions $q_n(x)$ and the functions $h_n(x^{(2)}) \equiv h_n(x, x)$. Here,

$$h_n(x^{(\alpha)}) = \sum_{\lambda \vdash n} \begin{pmatrix} \alpha \\ \lambda' \end{pmatrix} s_\lambda(x)$$

denotes a replicated S-function [34] and

$$\begin{pmatrix} X \\ \lambda \end{pmatrix} = \prod_{x \in \lambda} \frac{X - c(x)}{h(x)}$$

is the generalized binomial coefficient associated to a partition λ. In particular, for $\alpha = 2$, we have

$$h_n(x^{(2)}) = \sum_{2m+p=n} (p+1) s_{(m+p,m)}(x).$$

Now

$$\sum_{n=0}^{\infty} h_n(x^{(2)}) z^n = \exp\left(\left(\sum_{n \text{ even}} + \sum_{n \text{ odd}}\right) \frac{2}{n} p_n(x) z^n\right) = \left(\sum_{k=0}^{\infty} h_k(x^2) z^{2k}\right)\left(\sum_{l=0}^{\infty} q_l(x) z^l\right).$$

Hence

$$h_{2n}(x^{(2)}) = \sum_{j=0}^{n} h_{n-j}(x^2)\, q_{2j}(x) \qquad h_{2n+1}(x^{(2)}) = \sum_{j=0}^{n} h_{n-j}(x^2)\, q_{2j+1}(x).$$

Alternatively we can write this as

$$q_{2n}(x) = (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ h_2(x^{(2)}) & h_1(x^2) & 1 & 0 & \cdots & 0 \\ h_4(x^{(2)}) & h_2(x^2) & h_1(x^2) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ h_{2n-2}(x^{(2)}) & h_{n-1}(x^2) & h_{n-2}(x^2) & \cdots & h_1(x^2) & 1 \\ h_{2n}(x^{(2)}) & h_n(x^2) & h_{n-1}(x^2) & \cdots & h_2(x^2) & h_1(x^2) \end{vmatrix}$$

$$q_{2n+1}(x) = (-1)^n \begin{vmatrix} h_1(x^{(2)}) & 1 & 0 & \cdots & 0 & 0 \\ h_3(x^{(2)}) & h_1(x^2) & 1 & 0 & \cdots & 0 \\ h_5(x^{(2)}) & h_2(x^2) & h_1(x^2) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ h_{2n-1}(x^{(2)}) & h_{n-1}(x^2) & h_{n-2}(x^2) & \cdots & h_1(x^2) & 1 \\ h_{2n+1}(x^{(2)}) & h_n(x^2) & h_{n-1}(x^2) & \cdots & h_2(x^2) & h_1(x^2) \end{vmatrix}$$

$$h_n(x^2) = (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ h_2(x^{(2)}) & q_2(x) & 1 & 0 & \cdots & 0 \\ h_4(x^{(2)}) & q_4(x) & q_2(x) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ h_{2n-2}(x^{(2)}) & q_{2n-2}(x) & q_{2n-4}(x) & \cdots & q_2(x) & 1 \\ h_{2n}(x^{(2)}) & q_{2n}(x) & q_{2n-2}(x) & \cdots & q_4(x) & q_2(x) \end{vmatrix}$$

$$h_n(x^2) = \frac{(-1)^n}{[q_1(x)]^{n+1}} \begin{vmatrix} h_1(x^{(2)}) & q_1(x) & 0 & \cdots & 0 & 0 \\ h_3(x^{(2)}) & q_3(x) & q_1(x) & 0 & \cdots & 0 \\ h_5(x^{(2)}) & q_5(x) & q_3(x) & q_1(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ h_{2n-1}(x^{(2)}) & q_{2n-1}(x) & q_{2n-3}(x) & \cdots & q_3(x) & q_1(x) \\ h_{2n+1}(x^{(2)}) & q_{2n+1}(x) & q_{2n-1}(x) & \cdots & q_5(x) & q_3(x) \end{vmatrix} .$$

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